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On the Problem of the Spherical Representation and the Characteristic Equations of Certain Classes of Surfaces.[*]

By Archer Everett Young.

§ 1. *Introduction.*

If systems of lines on the Gauss sphere cutting at right angles are given, the determination of the most general class of surfaces having these as the spherical representation of their lines of curvature requires the integration of a Laplace equation. The problem of determining those systems of lines on the sphere for which the complete systems of surfaces corresponding can be obtained has been discussed by Darboux.[†] Treating the problem from a purely analytical standpoint, he reduces it to the determination of those Laplace equations having equal invariants which permit of particular solutions of modulus one, and which moreover can be integrated.[‡]

Considering the equations of this class, for which the sequence formed according to the method of Laplace terminates, and employing the analytic theory developed by Moutard,[§] Darboux shows that, from the solution of any one equation, can be obtained the solutions of all the others by simple quadratures.[||] From a purely analytical point of view, this treatment of the problem is most direct. The results, however, are not in such a form as to be easily incorporated with the other results of differential geometry, since the tangential coordinates alone are found, and since none of the functions appearing in the three fundamental forms for a surface are employed. Moreover, if one wishes to consider the isothermic surfaces which are solutions of the problem, it is impossible to do so from his treatment.[¶]

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[*] Read before the Amer. Math. Society, Chicago Section, April 18, 1908.

[†] Théorie des Surfaces, Vol. IV, p. 169.

[‡] *Loc. cit.*, p. 178.

[§] *Loc. cit.*, p. 181.

[||] *Loc. cit.*, p. 184.

[¶] Darboux realizes this fact, for he is obliged to use an entirely different treatment in determining the isothermic surfaces having one set of their lines of curvature plane. (See Vol. IV, Chapter X.)

We have considered the same problem in this paper from more of a geometric standpoint, taking as our point of departure the Gauss and Codazzi equations written in a form first used by Bonnet.[*] The functions involved are those which appear in the first and third form for a surface, together with two others which are closely related to them. Particular cases can be treated with great facility, and the characteristic forms for the linear elements may be written out immediately.[†] The systems of isothermic surfaces which are solutions of the problem, following this method, are easily treated by themselves.

We begin in §2 by explaining the meaning of the expression "*characteristic equation*" as used in this paper. In §3, we consider analytically the general class of surfaces which are solutions of the problem of the spherical representation. We have made use of the general theory as developed by Darboux[‡] in order to show the relation between the systems of surfaces corresponding to particular solutions of the general characterizing equation.

In §4, we have called attention to a large class of surfaces belonging to the problem of the spherical representation which have their linear elements in a particular form.

The isothermic surfaces have been treated in §5, where it is shown that the general class may be obtained by simple algebraic processes combined with the integration of equations which are linear in a function of u and a function of v. In particular, we have pointed out that all isothermic surfaces having one set of their lines of curvature spherical can be obtained in this way from isothermic surfaces having one set of their lines of curvature plane.

In §6 and §7, we have considered the surfaces characterized respectively by two equations which play an important rôle in the general discussion. Among these, are the general class having one set of their lines of curvature plane or spherical. An important fact which we have noted in the first of these two articles — namely, that all isothermic surfaces, having one set of their lines of curvature plane, must have the same spherical representation of their lines of curvature as the Bonnet surfaces — makes unnecessary the discussion of these surfaces, to which Darboux has devoted the tenth chapter of volume four of the Théorie des Surfaces.

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[*] See reference to Bonnet's work given below.

[†] See §6 and §7.

[‡] See previous references.

§ 2. *Characteristic Equations of Certain Classes of Surfaces.*

The Gauss and Codazzi equations for a surface referred to lines of curvature may be written as follows:

$$\left.\begin{aligned} &\frac{\partial M}{\partial v}-\frac{\partial N}{\partial u}-PQ=0,\\ &\frac{\partial P}{\partial v}+MQ=0,\\ &\frac{\partial Q}{\partial u}-NP=0,\end{aligned}\right\} \qquad (1)^*$$

where

$$M=-\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v},\quad N=\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u},\qquad (2)$$

and where the first and third forms for the surfaces are respectively

$$\overline{ds}^2=E\overline{du}^2+G\overline{dv}^2,\qquad (\mathrm{I})$$

$$\overline{d\sigma}^2=P^2\overline{du}^2+Q^2\overline{dv}^2.\qquad (\mathrm{III})$$

In the system (1) are three equations involving four functions. If a fourth equation involving one or more of these functions is added, a certain system of lines on the Gauss sphere, in general, will be determined. We shall say that the equation added *defines the lines on the sphere*, and moreover, since only certain surfaces may have these as the spherical representation of their lines of curvature, we shall say that it is *characteristic of these surfaces.*

1°. For example, let us take with the system (1) the equation

$$M=0.\qquad (3)$$

The linear element on the Gauss sphere takes the form

$$\overline{d\sigma}^2=\overline{du}^2+\phi(u,v)\,\overline{dv}^2,$$

where the form of the function ϕ is easily determined. Hence the lines $v=$ const. are geodesics or great circles. We say that equation (3) *defines on the sphere all systems composed of great circles and their orthogonal trajectories.*

Moreover, the first form for the corresponding surfaces may be written

$$\overline{ds}^2=du^2+\psi(u,v)\,dv^2,$$

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[*] Bonnet's form for these equations. Darboux sometimes uses them in this form, but not in the problem of the spherical representation.

where ψ has a certain form. Hence the lines $v =$ const. on these are geodesics. We say therefore that (3) is *characteristic of those surfaces which have one set of their lines of curvature geodesics.* Geodesic lines of curvature are of course plane lines.

2°. If the equation

$$M + N = 0 \tag{4}$$

is used with the system (1), a very general system of lines is defined on the Gauss sphere. The first and third forms for the surfaces characterized by this equation may be written thus:*

$$\overline{ds}^2 = \frac{\partial \phi}{\partial u} du^2 + \frac{\partial \phi}{\partial v} dv^2, \tag{I}$$

$$\overline{d\sigma}^2 = \frac{\partial \psi}{\partial u} du^2 + \frac{\partial \psi}{\partial v} dv^2, \tag{III}$$

where ϕ and ψ are functions of u and v which must be chosen so as to satisfy the Gauss and Codazzi equations.

Belonging to this very general class of surfaces, we shall in a later article, § 4, call attention to a large system which is a solution of the problem of the spherical representation.

3°. The equation

$$\frac{\partial M}{\partial u} + \frac{\partial N}{\partial v} = 0, \tag{5}$$

used with (1), defines upon the Gauss sphere *those systems of lines which permit of isothermic surfaces having them as the spherical representation of their lines of curvature.* For the linear element of any isothermic surface, if the parameters be properly chosen, may be written thus:

$$\overline{ds}^2 = \lambda (du^2 + dv^2),$$

and from (2) it appears then that (5) is satisfied. The equation (5) plays an important rôle in the discussion of the isothermic surfaces which are solutions of the problem of the spherical representation. †

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* For $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u}$.

† See § 5.

§ 3. *The Problem of the Spherical Representation.*

Of the defining equations noted in the preceding article, only the first, $M = 0$, enables one to obtain the *general solution* of the systems of equations (1) and (2). There are, however, infinitely many equations which permit of this, as we may see as follows.

Combining the last two equations of (1) and solving the second for Q, we have, in the place of this system,

$$\left.\begin{aligned} &\frac{\partial M}{\partial v} - \frac{\partial N}{\partial u} - PQ = 0, \\ &\frac{\partial^2 P}{\partial u \partial v} - \frac{\partial \log M}{\partial u}\frac{\partial P}{\partial v} + MNP = 0, \\ &Q = -\frac{\frac{\partial P}{\partial v}}{M}. \end{aligned}\right\} \qquad (6)$$

The system (2), likewise, may be replaced by

$$\left.\begin{aligned} &\frac{\partial^2 \sqrt{E}}{\partial u \partial v} - \frac{\partial \log M}{\partial u}\frac{\partial \sqrt{E}}{\partial v} + MN\sqrt{E} = 0, \\ &\sqrt{G} = -\frac{\frac{\partial \sqrt{E}}{\partial v}}{M}. \end{aligned}\right\} \qquad (7)$$

The second equation in (6) is a Laplace equation in P. If M and N have such values that this equation can be solved, the corresponding solution of the system, as we shall see, can be obtained. Such a solution determines the lines on the Gauss sphere, and the linear element of the general class of surfaces having these as the spherical representation of their lines of curvature, can be obtained at once from (7).*

The whole problem then hinges upon the solution of a Laplace equation, the general solution of which, excluding a few particular cases, can be obtained when, and only when, one or both of the sequences, formed for it according to the method of Laplace, terminates; that is, leads to an equation for which one of the invariants vanishes.†

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* The Laplace equations in (6) and (7) are of course the same, but the solution of (6) is conditioned by the Gauss equation, whereas that of (7) is unconditioned.

† Darboux, *loc. cit.*, Vol. II, p. 23.

Calling the invariants of the Laplace equation in (6) h and k,* we have:

$$\begin{aligned} h &= -MN, \\ k &= -\frac{\partial^2 \log M}{\partial u\, \partial v} - MN, \\ k_1 &= -\frac{\partial^2 \log (Mk)}{\partial u\, \partial v} - k, \\ k_2 &= -\frac{\partial^2 \log (Mk k_1)}{\partial u\, \partial v} - k_1, \\ &\ldots\ldots\ldots\ldots\ldots\ldots, \\ k_{i+1} &= -\frac{\partial^2 \log (Mk \ldots . k_i)}{\partial u\, dv} - k_i, \end{aligned}$$

where k_i is the principal invariant for the corresponding equation of the sequence formed for the original equation in the one way.

The equation obtained by equating to zero any one of these invariants, taken with (6), *defines a system of lines on the Gauss sphere, and characterizes the general class of surfaces having these as the spherical representation of their lines of curvature.*†

Suppose we take as the defining equation

$$k_{i+1} = 0, \tag{8}$$

where i is arbitrarily large. Assuming that M and N have such values that (8) is satisfied, we can obtain P and Q from the last two equations of (6) at once in terms of M and N and arbitrary functions of integration.

Substituting in the first of (6), we have an equation in M and N which, used with (8), will determine them.

The values of E and G, which may now be found directly from (7), make known the linear element of the general class of surfaces. It is quite clear, however, that the actual determination of the surfaces in this way for any large value of i would be a very difficult, if not impossible problem. We shall point out, therefore, *a method for determining the general class of surfaces defined by* (8) *from those characterized by* $M = 0$.‡

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* Darboux, *loc. cit.*, Vol. II, p. 30.

† The problem of the spherical representation as we have considered it in this paper has to do with such surfaces only.

‡ Similar to Darboux's results as geometrically interpreted by him (see *loc. cit.*, Vol. IV, p. 181).

It is clear that the lines on the sphere defined by $M=0$[*] are included as a particular case in the class defined by $k=0$, and that those defined by the latter are included as a particular case in the class defined by $k_1=0$. In general, then, *the lines defined by* $k_i=0$ *are a part of the class defined by* $k_{i+1}=0$, *and hence the surfaces characterized by the former are included in the class characterized by the latter equation.*

It follows from Darboux's work[†] that *the lines defined by* $k_{i+1}=0$, *which are not included in the class defined by* $k_i=0$, *are the spherical representation of the lines of curvature of the inverse with regard to a sphere of certain surfaces defined by the latter equation.* Hence the surfaces characterized by $k_{i+1}=0$, which are not included among those characterized by $k_i=0$, are *the inverse of certain of these*, or *surfaces which have the same spherical representation of their lines of curvature as these surfaces obtained by inversion.* To obtain, therefore, the general class of surfaces characterized by (8) we proceed as follows.

Taking the inverse of the general class defined by $M=0$, we obtain by algebraic methods[‡] a new set of values for P and Q, and hence of M and N by (2). This new set of values is the general solution of (6) and $k_1=0$.[§] The linear element of the most general class of surfaces characterized by $k=0$ is obtained directly by the integration of (7). Since the direction cosines of the normal to the surfaces (that is, the Cartesian coordinates X, Y, Z of a point on the Gauss sphere) are known, and since the principal radii of curvature r_1, r_2 can be obtained by division, the Cartesian coordinates for the surfaces characterized by $k=0$ are given by the integration of the following equations:

$$\frac{\partial x}{\partial u}=r_2\frac{\partial X}{du},\quad \frac{\partial x}{\partial v}=r_1\frac{\partial X}{\partial v},\ \text{etc.}$$

Taking the inverse of the surfaces thus found and repeating the operation, we obtain the Cartesian coordinates of the general class characterized by $k_1=0$. Repetition, finally, leads to the surfaces characterized by (8) for any value of i.

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[*] The equation $h=0$ may be replaced by $M=0$ without geometrical limitation, evidently.

[†] See previous references, and in particular Vol. IV, p. 181.

[‡] We are assuming that the Cartesian coordinates of the surfaces characterized by $M=0$ have been obtained. See also Darboux, *loc. cit.*, Vol. II, p. 247.

[§] Must be, according to Darboux's work. See references above.

§ 4. *Surfaces of the Problem of the Spherical Representation Characterized by* $M + N = 0$.

In the preceding article we assumed that the Laplace equation in (6) was such that the sequence formed in the one way was finite. Had we assumed that both sequences were finite (that is, that M and N had such values that, for some value of i, $h_i = k_i = 0$), we would have found that the ratio of M to N must be a function of u over a function of v, or choosing parameters properly,

$$M + N = 0.$$

Hence, writing $-M$ in place of N in (6), we have as the defining equation to be used with it

$$\frac{\partial^2 \log (M k k_1 \dots . k_i)}{\partial u \, \partial v} + k_i = 0, \tag{8'}$$

where $k k_1 \dots . k_i$ are defined as in § 3.

We gave in § 2 the general form which the linear element of the surfaces characterized by (8′) takes. The surfaces obtained by inversion from a given class of these do not in general belong to the general class defined by (8′), but to the more general class characterized by (8). We can easily determine the necessary form* of the expression for the linear element of isothermic surfaces included here. For from § 2 it appears that

$$\frac{\dfrac{\partial \phi}{\partial u}}{\dfrac{\partial \phi}{\partial v}} = \frac{f(u)}{\phi(v)},$$

and hence, by a change of parameters,

$$\overline{ds}^2 = F(u + v) \left(\frac{du^2}{U} + \frac{dv^2}{V} \right).$$

§ 5. *Isothermic Surfaces.*

We found in § 2 that if the lines on the Gauss sphere are to permit of isothermic surfaces, then for a proper choice of parameters M and N must satisfy the equation (5). *The isothermic surfaces belonging to the problem of the*

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[*] This form is not characteristic of these surfaces. The linear element for the quadrics comes under the same form. The quadrics, however, might be considered a solution of the problem, since the linear element for the general class of surfaces having the same spherical representation of their lines of curvature can be obtained, as the Laplace equation can be integrated, though not by the method of Laplace.

spherical representation can be determined easily by algebraic operations and simple integration from those characterized by $M=0$. To prove this, let us note first that *all isothermic surfaces belonging to the class characterized by* $k_{i+1}=0$, *which do not at the same time belong to the class of surfaces characterized by* $k_i=0$, *must have the same spherical representation of their lines of curvature as the inverse of some isothermic surface* belonging to* $k_i=0$. For all lines defined by $k_{i+1}=0$, which do not also belong to $k_i=0$, are the spherical representation of the lines of curvature of *some* surface belonging to the class characterized by the latter; and for any surface *other than an isothermic surface* this would require that a function θ already uniquely determined as follows:

$$\theta=[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]^2,$$

where x, y, z and α, β, γ are respectively the coordinates of a point on the surface which is to be inverted, and of the pole of inversion, should necessarily be a solution of the equation†

$$\frac{\partial}{\partial v}\left(\frac{\theta}{\sqrt{E}}\,\frac{\partial\left(\frac{\sqrt{G}}{\theta}\right)}{\partial u}\right)-\frac{\partial}{\partial u}\left(\frac{\theta}{\sqrt{G}}\,\frac{\partial\left(\frac{\sqrt{E}}{\theta}\right)}{\partial v}\right)=0, \tag{9}$$

where E and G are the functions in the expression for the linear element of the original surfaces.‡

Hence, to obtain the isothermic surfaces included in the class characterized by $k_{i+1}=0$, we begin with the isothermic surfaces included in the general class of surfaces $M=0$. These are surfaces of revolution and certain developable surfaces.

Taking their inverse with regard to a sphere, we have the "Bonnet surfaces",§ which of course belong to the class characterized by $k=0$. If we let the linear element of all other isothermic surfaces belonging to this class be written in the form

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* The inverse of an isothermic surface is of course an isothermic surface, and hence the corresponding lines on the sphere are defined by (5) and permit of such.

† See equation (5) and latter part of § 2.

‡ Of course it is possible that such a relation does exist in a very particular case; but it is easily shown that it does not exist for any known surfaces, and it is quite probable that there is no exception to the rule. If $E=G$, as is the case for isothermic surfaces, then (9) vanishes identically, irrespective of the values of the functions.

§ A name given to these surfaces because they were first determined by Bonnet. (*Journal de l'École Polytechnique*, Vol. XLII, pp. 132–151.)

$$\overline{ds}^2 = \lambda\left(\frac{du^2}{U} + \frac{dv^2}{V}\right),$$

where λ is a function of u and v to be determined, it follows that

$$M = -\sqrt{\frac{V}{U}}\frac{\partial \log \sqrt{\lambda}}{\partial v}, \quad N = \sqrt{\frac{U}{V}}\frac{\partial \log \sqrt{\lambda}}{\partial u},$$

where *M and N are formed for the Bonnet surfaces and hence are known.* The condition for integrability gives

$$\frac{\partial}{\partial u}\left(M\sqrt{\frac{U}{V}}\right) + \frac{\partial}{\partial v}\left(N\sqrt{\frac{V}{U}}\right) = 0, \tag{10}$$

an equation which is linear in U and V, and hence can in general be integrated.* The values of these functions being thus determined, the function λ is obtained immediately by quadrature.

As the direction cosines of the normal for the Bonnet surfaces are known, and as the principal radii of curvature r_1 and r_2 are obtained by division from the functions found, it follows that the Cartesian coordinates for the isothermic surfaces characterized by $k = 0$ are given at once by quadrature.† Beginning with the class of surfaces just determined and proceeding exactly as before, we obtain the isothermic surfaces‡ characterized by $k_1 = 0$, and from these those characterized by $k_2 = 0$, and so on, till finally we come to those characterized by $k_{i+1} = 0$ whatever the value of i.

This general method for determining the isothermic surfaces connected with the problem of the spherical representation has not, I believe, been pointed out before. The only difficulty to be met with is in the integration of (10) for the different values of M and N. An immediate application of the theory§ would be in the determination of all isothermic surfaces which have one set of their lines of curvature spherical. These, as is easily shown (see § 6 and § 7), have the same spherical representation of their lines of curvature as the inverse of the isothermic surfaces having one set of their lines of curvature plane.

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* We have already determined these surfaces by this method in a paper "On a Certain Class of Isothermic Surfaces" (*Transactions of the Amer. Math. Society*, Vol. X, pp. 79–94).

† See latter part of § 3.

‡ These include as a particular case all isothermic surfaces having one set of their lines of curvature spherical.

§ This theory will apply whether the isothermic surfaces are connected with the problem of the spherical representation proper or not. For suppose we have given any class of isothermic surfaces. The integration of (10) will give *all* isothermic surfaces having the same spherical representation, and inversion gives a new class with new lines on the sphere. The integration of (10) again applied to the new class would in general lead to a still more general class, etc.

§ 6. *Surfaces Characterized by $k = 0$.*

As we have already pointed out, *all surfaces characterized by the equation*

$$\frac{\partial^2 \log M}{\partial u\, \partial v} + MN = 0 \tag{11}$$

have the same spherical representation of their lines of curvature as the surfaces characterized by $M = 0$, and the inverse of these. Integrating, we find, from (6) and (7),* that the first and third forms for these surfaces may be written thus:

$$\overline{ds}^2 = M^2\left[\int \frac{f(u)\,du}{M} + \phi(v)\right]^2 du^2 + \frac{1}{M^2}\left[\frac{\partial}{\partial v}\left(M\int \frac{f(u)\,du}{M} + \phi(v)\right)\right]^2 dv^2, \tag{I}$$

$$\overline{d\sigma}^2 = M^2\left[\int \frac{U\,du}{M} + V\right]^2 du^2 + \frac{1}{M^2}\left[\frac{\partial}{\partial v}\left(M\int \frac{U\,du}{M} + V\right)\right]^2 dv^2, \tag{III}$$

where in (I) M, and hence the lines on the sphere, are entirely independent of the arbitrary functions† of integration $f(u)$ and $\phi(v)$, but in (III) M is a solution of an equation (the Gauss equation) in which the functions U and V appear.

The lines $v = \text{const.}$ on the surfaces $M = 0$ (see § 2) are plane lines, and the system of planes in which the lines lie cuts the surfaces at right angles.‡ The lines $v = \text{const.}$ on the inverse of these surfaces, therefore, will be, in general, spherical lines, and, moreover, lines which have constant geodesic curvature, since the systems of planes will invert into systems of spheres cutting the surfaces at the original angle; that is, at right angles.

The necessary and sufficient condition that the lines on the surfaces corresponding to (I) *shall have constant geodesic curvature along the lines $v = \text{const.}$ is, that the function $f(u)$ shall vanish.*§ The value of this function, however, does not affect the lines on the sphere, as we have already noted; ‖ and hence, as we would expect from the general theory, *all the surfaces characterized by* (11) *have the same spherical representation of their lines of curvature as those having constant geodesic curvature along the lines $v = \text{const.}$*

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* Assuming that M and N satisfy (11).

† These are the functions appearing in the general integral of the Laplace equation in (7).

‡ A line of curvature having constant geodesic curvature, not zero, is the intersection of the surface and a sphere cutting at right angles. If the curvature is zero, the line is a geodesic line and the sphere is a plane.

§ For $\frac{M}{\sqrt{E}}$ must equal a function of v. (See Bianchi, Vol. I, p. 210.)

‖ See second note to § 3. This is merely an arbitrary function of the integral of the Laplace equation in (7).

It does not follow by any means that all the surfaces having constant geodesic curvature along the lines $v=$ const. are obtained by inversion from those characterized by $M=0$, but merely that they all have the same spherical representation of their lines of curvature as these.

Now, it is easily shown* that all surfaces having one set of the lines of curvature plane are included in the class (11); and since the lines on the sphere which are the spherical representation of the lines of curvature must also be plane, it appears that in (III), for this particular class, U must vanish,† and hence the form (I), as written, and (III), with $U=0$, are characteristic of these surfaces.

We can now state the theorem, *that all isothermic surfaces having one set of their lines of curvature plane must have the same spherical representation of their lines of curvature as the Bonnet surfaces.* We have already proved this theorem, as we showed above that all isothermic surfaces characterized by (11) have the same spherical representation of their lines of curvature as the Bonnet surfaces,‡ and as we have now just pointed out that all surfaces having one set of the lines of curvature plane are included in the system (11). This theorem reduces to one of a very elementary nature, the *problem of determining all isothermic surfaces having one set of the lines of curvature plane,* a problem which was solved in its most general form by Darboux,§ a particular case being solved later by P. Adam.‖ As we shall show¶ (§7) that all isothermic surfaces having one set of their lines of curvature spherical can be obtained from the isothermic surfaces having one set plane, the determination of these latter surfaces has been made a somewhat easier problem.

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* Expressing the fact that the lines on the Gauss sphere must have constant geodesic curvature, we find that M and N must satisfy (11).

† For $\frac{M}{Q}$ must equal a function of v.

‡ We might prove this directly as follows. Differentiating the equation formed by expressing the fact that the expression (I) corresponds to isothermic surfaces and holding count of the fact at the same time that M, and hence the lines on the sphere, must be independent of the function $f(u)$, we find that, choosing parameters properly, $M=-\sqrt{\frac{V_1}{U_1}}\frac{1}{(u+v)}$; and $N=\sqrt{\frac{U_1}{V_1}}\frac{1}{(u+v)}$, using (11). But the only lines on the sphere corresponding to these values of M and N are those which are the spherical representation of the lines of curvature of the Bonnet surfaces. (See Bonnet, *loc. cit.*, p. 132.)

§ *Loc. cit.*, Vol. IV, Chapter X.

‖ *Annales scientifiques de l'École Normale supérieure*, 3ᵉ série, t. X, p. 319, 1893.

¶ This follows almost directly from the relation shown between surfaces having plane and spherical lines and from the work of §5.

Before closing this article, let us note that the only surfaces having one set of their lines of curvature spherical, belonging to (11), are those which have constant geodesic curvature along them.

§ 7. *Surfaces Characterized by* $k_1 = 0$.

The surfaces characterized by (11) may be divided into three classes as follows:

1°. Those having one set of their lines of curvature plane (corresponding to (11), when $U = 0$).

2°. Those having constant geodesic curvature along one set of the lines of curvature (corresponding to (11), when $f(u) = 0$ and $U \neq 0$).

3°. Surfaces having the same spherical representation of their lines of curvature as those in 2° (corresponding to (11), when $f(u) \neq 0$ and $U \neq 0$).

We shall now consider the surfaces obtained by inversion from these three classes, and the other surfaces which have the same spherical representation of their lines of curvature as these. As a whole, they would be characterized by the equation $k_1 = 0$, or

$$\frac{\partial^2 \log (Mk)}{\partial u \, \partial v} + k = 0, \tag{12}$$

where $k = -\dfrac{\partial^2 \log M}{\partial u \, \partial v} - MN$. Their first and third forms by (6) and (7), holding account of (12), may be written thus:

$$\begin{aligned} \overline{ds}^2 = M^2 \left[\int k \left(\int \frac{f(u)}{Mk} du + \phi_1(v) \right) du + \phi_2(v) \right]^2 du^2 \\ + \frac{1}{M^2} \left[\frac{\partial}{\partial v} \left\{ M \left[\int k \left(\int \frac{f(u)}{Mk} du + \phi_1(v) \right) du + \phi_2(v) \right] \right\} \right]^2 dv^2, \end{aligned} \tag{I}$$

$$\begin{aligned} \overline{d\sigma}^2 = M^2 \left[\int k \left(\int \frac{U du}{Mk} + V_1 \right) du + V_2 \right]^2 du^2 \\ + \frac{1}{M^2} \left[\frac{\partial}{\partial v} \left\{ M \left[\int k \left(\int \frac{U du}{Mk} + V_1 \right) du + V_2 \right] \right\} \right]^2 dv^2, \end{aligned} \tag{III}$$

where, as before, M and also k are independent of the functions appearing in (I), but dependent upon those appearing in (III), since the latter appear in an equation which M and k must satisfy. The surfaces obtained by inversion from the class 1° will, in general, have the lines $v =$ const. spherical. These belong to the class characterized by (12),* when $k \neq 0$. The surfaces which are the inverse of 2° all belong to the class characterized by (11), for either they must have the lines $v =$ const. plane and hence belong to the class (11) or else they have constant geodesic curvature and belong to the same.

It appears, therefore, *that the only surfaces characterized by* (12), *when* $k \neq 0$, *which have one set of their lines of curvature spherical are the inverse of* 1°, *together with certain surfaces having the same spherical representation of their lines of curvature.*† We can easily show to what form (I) and (III) reduce for surfaces having one set of their lines of curvature spherical. For Bianchi ‡ has shown that the necessary and sufficient condition that the lines $v =$ const. shall be spherical is that the principal radius of curvature r_2 shall be defined as follows:

$$r_2 = \alpha + \frac{\beta}{PQ}\frac{\partial P}{\partial v},$$

where α and β are functions of v. Applying this condition to the expression for r_2 § obtained from (I) and (III), we can easily show by differentiation that both the functions $f(u)$ and U must vanish. Hence (III), with $U = 0$, corresponds to lines on the sphere which permit of surfaces having the lines $v =$ const. spherical. Moreover, *this is the linear element on the sphere for all surfaces having one set of their lines of curvature spherical, since the equation* $k_2 = 0$ *characterizes no surfaces having spherical lines* (*unless* $k_1 = 0$).

It follows, therefore, *that all surfaces having one set of their lines of curvature spherical have the same spherical representation of their lines of curvature as the inverse of the general class having one set of their lines of curvature plane.* This is a theorem which has been proved before in an entirely different way. ||

Collecting results, we see that the surfaces characterized by (12) and corresponding to (I) and (III) may be classified as follows:

1°. Surfaces having the lines $v =$ const. spherical (corresponding to the case where $f(u) = U = 0$).

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[*] Having spherical lines, they would not belong to (11), for they could not have constant geodesic curvature along them, since the spheres into which the planes invert would cut the surfaces at the original angle, which could not be 90° unless $M = 0$.

[†] Darboux has discussed surfaces having one set of their lines of curvature spherical (*loc. cit.*, Vol. IV, Chapter XI), following his general treatment of the problem of the spherical representation. Our results, so far as they can be compared, are in full agreement with his.

[‡] *Loc. cit.*, p. 304.

[§] $r_2 = \frac{\sqrt{E}}{P}$.

[||] Darboux, *loc. cit.*, Vol. IV, p. 252.

2°. Surfaces having the same spherical representation of their lines of curvature as those of the preceding class (corresponding to the case $U = 0$ and $f(u) \neq 0$).

3°. Surfaces which are the inverse of the third class characterized by (11), and the others which have the same spherical representation of their lines of curvature as these (corresponding to the general case where $f(u) \neq 0$, $U \neq 0$).

The inverse of class 1° would give no new surfaces, but the inverse of 2°, and of 3° in part, would lead to other surfaces characterized by $k_2 = 0$. The general class of surfaces characterized by this equation is composed of these, of course, and the others having the same spherical representation of their lines of curvature.

In closing this paper, we will call attention to the ease with which the question as to whether any known surfaces are solutions of the problem of the spherical representation can be settled. If either the first or third forms are known, the functions M and N can be formed immediately, and substitution in the equation $k_{i+1} = 0$ settles the question. From Darboux's treatment of the problem such a question could be answered only after the greatest difficulty.

MIAMI UNIVERSITY.